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Fourier frames

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Abstract

We solve the problem of Duffin and Schaeffer (1952) of characterizing those sequences of real frequencies which generate Fourier frames. Equivalently, we characterize the sampling sequences for the Paley-Wiener space. The key step is to connect the problem with de Branges' theory of Hilbert spaces of entire functions. We show that our description of sampling sequences permits us to obtain a classical inequality of H. Landau as a consequence of Pavlov's description of Riesz bases of complex exponentials and the John-Nirenberg theorem. Finally, we discuss how to transform our description into a working condition by relating it to an approximation problem for subharmonic functions. By this approach, we determine the critical growth rate of a nondecreasing function ψ such that the sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ defined by $\lambda_k + \psi(\lambda_k) = k$ is sampling.

1. Introduction

Following Duffin and Schaeffer [DS52], we say that a system of complex exponentials $\{e^{i\lambda_k x}\}$, with $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ a sequence of real numbers, is a *Fourier frame* if there exist positive constants A and B such that

$$A\int_{-\pi}^{\pi}|f(x)|^2dx \le \sum_{k=-\infty}^{\infty}\left|\int_{-\pi}^{\pi}f(x)e^{-i\lambda_k x}dx\right|^2 \le B\int_{-\pi}^{\pi}|f(x)|^2dx$$

for all $f \in L^2(-\pi,\pi)$. The purpose of this paper is to give a description of those sequences Λ that generate Fourier frames.

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The main result of [DS52] is a sufficient density condition for $\{e^{i\lambda_k x}\}$ to constitute a Fourier frame. Over the past two decades, the work of Duffin and Schaeffer has been remarkably influential, but mainly so because of its description of abstract frame expansions as an alternative to orthonormal bases; see, e.g., [Da92]. There has been little progress on the problem of characterizing the original Fourier frames of Duffin and Schaeffer.

Our work relies on several profound results. Specifically, it may be seen as an interplay between three themes:

1) *de Branges' Hilbert spaces of entire functions.* Our main theorem is based on de Branges' theory [dB68]. The Hilbert space of interest to us is the classical Paley-Wiener space, the prime example of a de Branges space. We denote this space by PW; it consists of all entire functions of exponential type at most π whose restrictions to $\mathbb{R}$ are square-integrable. We say that a sequence of real numbers $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ is *sampling* for PW if there exist positive constants A and B such that

$$A\int_{-\infty}^{\infty} |f(x)|^2\,dx \le \sum_{k=-\infty}^{\infty} |f(\lambda_k)|^2 \le B\int_{-\infty}^{\infty} |f(x)|^2\,dx$$

holds for all $f \in \mathrm{PW}$. By the Paley-Wiener theorem, Λ *is sampling for* PW *if and only if* $\{e^{i\lambda_k x}\}$ *is a Fourier frame.* In other words, our task is to describe the sampling sequences for PW.

The Paley-Wiener space is a Hilbert space when equipped with its standard norm, the $L^2(\mathbb{R})$-norm. The symbol PW will stand for this particular Hilbert space. However, if Λ is a sampling sequence, we may equip the Paley-Wiener space with an equivalent norm given by $\sqrt{\sum_k |f(\lambda_k)|^2}$. In this way, we obtain a different Hilbert space, which is again a de Branges space. This simple observation has the amazing consequence that several basic results of de Branges' theory apply to our problem. In particular, two results of de Branges about *norm identities* may be combined to yield a result about *norm equivalence* (Theorem 1 below).

The required background on de Branges spaces is reviewed in Section 2, and then our main theorem — a necessary and sufficient condition for sampling — is stated and proved in Section 3.

2) *Riesz bases of complex exponentials.* Fourier frames are intimately connected with Riesz bases of complex exponentials, or equivalently, sampling sequences are intimately connected with complete interpolating sequences (see below for definition). This is of interest to us, because there exists a beautiful description of the latter kind of sequences, due to Pavlov [Pa79]; see [HNP81] for an extensive account and [LS97] for some recent progress on this topic.

We say that Λ is a *complete interpolating sequence* for PW if the interpolation problem $f(\lambda_k) = a_k$ has a unique solution $f \in \mathrm{PW}$ whenever $\{a_k\}$ is square-summable, or equivalently, if it is sampling but fails to be so on the removal of any one of the points λ_k. Thus a complete interpolating sequence is a sampling sequence with no "redundant" points. The sequence of integers is of course the leading example of such a sequence.

Some additional terminology is required to state Pavlov's theorem. Let us assume that $\lambda_k \leq \lambda_{k+1}$ for all k. A sequence Λ is *separated* if

$$q = \inf_k (\lambda_{k+1} - \lambda_k) > 0;$$

q is referred to as the *separation constant* of Λ. With a separated sequence Λ we associate a distribution function $n_\Lambda(t)$ defined such that for $a < b$

$$n_\Lambda(b) - n_\Lambda(a) = \#(\Lambda \cap (a, b]),$$

and normalized such that $n_\Lambda(0) = 0$. There is clearly a one-to-one correspondence between Λ and n_Λ. It is plain that all complete interpolating sequences are separated.

Let

$$P_y[h](x) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{h(t)}{(x-t)^2 + y^2}\, dt$$

denote the Poisson extension of h to the upper half-plane $y > 0$, and let $u \mapsto \tilde{u}$ stand for the usual conjugation operator. The following theorem, with the Helson-Szegö condition appearing in slight disguise, is a reformulation (see [HNP81, p. 286]) of Pavlov's original result.

Theorem A (Pavlov; Hruščev, Nikol'skiĭ, Pavlov). *A separated sequence Λ of real numbers is a complete interpolating sequence for* PW *if and only if $h(t) = n_\Lambda(t) - t$ is a function in* $\mathrm{BMO}(\mathbb{R})$ *such that $P_1[h](x) = \tilde{u}(x) + v(x) + C$, with $u, v \in L^\infty(\mathbb{R})$ and $\|v\|_\infty < 1/4$.*

To see the problem of relating this result to sampling sequences, we begin by stating two density conditions for sampling. The first is a fairly elementary result: Λ *is sampling if there exist positive constants ε and C such that*

$$n_\Lambda(b) - n_\Lambda(a) \geq (1+\varepsilon)(b-a) - C$$

for all $a < b$. This is essentially what was obtained by Duffin and Schaeffer (see also Theorem 2.1 of [Se95]). A more profound result is the following inequality[1] of Landau [La67].

[1] In [La67], the inequality is given in the following more general form. Let Ω be a finite union of intervals with $|\Omega| = 2\pi$ and $\mathrm{PW}(\Omega)$ the subspace of $L^2(\mathbb{R})$ consisting of functions whose Fourier transforms vanish outside Ω; define sampling sequences for $\mathrm{PW}(\Omega)$ as above. Then the inequality remains true if we replace PW by $\mathrm{PW}(\Omega)$.

Theorem B (Landau). *If* Λ *is a separated sampling sequence for* PW, *then*

$$n_\Lambda(b) - n_\Lambda(a) \geq b - a - A\log^+(b-a) - B$$

when $a < b$, *with constants* A *and* B *independent of* a, b.

Landau's inequality is best possible, as shown by an example in Section 4. We shall see that Theorem A and our main theorem (Theorem 1) imply that *Landau's inequality is a consequence of the John-Nirenberg theorem for* BMO *functions.* This result will be proved in Section 4 below.

The two inequalities just recorded have the following consequence. Set

$$D^-(\Lambda) = \lim_{R\to\infty} \frac{\min_x(n_\Lambda(x+R) - n_\Lambda(x))}{R},$$

which is the Beurling lower uniform density of Λ. Then Λ is sampling if $D^-(\Lambda) > 1$ and fails to be sampling if $D^-(\Lambda) < 1$. (See also [Ja91] for a somewhat different formulation.) It was proved in [Se95] that when $D^-(\Lambda) > 1$, there exists a subsequence $\Lambda' \subset \Lambda$ such that Λ' is a complete interpolating sequence, but an example was given of a sampling sequence Λ for which no subsequence is a complete interpolating sequence. *The latter type of sampling sequences are the only ones which are essentially different from complete interpolating sequences.* The third theme of this paper is what is needed to explore their subtle properties.

3) *Approximation of subharmonic functions.* We have in mind the following general type of problem. Given a subharmonic function U, find an entire function f such that $\log|f|$ is in some sense a good approximation to U. Our main theorem becomes a working condition only if we are able to solve certain problems of this kind. An exhaustive discussion of the solutions relevant for our sampling condition seems impracticable. Instead, we use an approach of Lyubarskii and Malinnikova [LM01] to elaborate an illustrative example. In Section 5, we determine the critical growth rate of a nondecreasing function ψ such that $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ defined by $\lambda_k + \psi(\lambda_k) = k$ is sampling. Roughly speaking, we obtain that if $\psi(x) = 0$ for $x < 0$, ψ is sufficiently regular, and $\psi(x) \to \infty$ when $x \to \infty$, then Λ is sampling if and only if $\psi(x)$ tends to ∞ at least as fast as $\log^+ x$.

This result may seem surprising: A sequence is sampling if and only if it constitutes a sufficiently large deviation from $\mathbb{Z}$. The central point here is that the sampling condition implies that we need to solve the above approximation problem for the logarithmic potential of the measure $d\psi$. Then difficulties arise if the measure is too much "spread out".

2. Preliminaries on de Branges spaces

A Hilbert space H of entire functions is a *de Branges space* if the following conditions are met:

(H1) Whenever f is in the space and has a nonreal zero ζ, the function $g(z) = f(z)(z-\bar{\zeta})/(z-\zeta)$ is in the space and has the same norm as f.

(H2) For every nonreal ζ the linear functional defined on the space by $f \mapsto f(\zeta)$ is continuous.

(H3) The function $f^*(z) = \overline{f(\bar{z})}$ belongs to the space whenever f belongs to the space and f^* has the same norm as f.

The simplest model example of a de Branges space is PW. A wider class of examples can be constructed via functions from the Hermite-Biehler class of entire functions. We denote this class by $\overline{\mathrm{HB}}$; it consists of all entire functions E with no zeros in the upper half-plane and satisfying $|E(z)| \geq |E(\overline{z})|$ whenever $\operatorname{Im} z > 0$. To every function $E \in \overline{\mathrm{HB}}$ we associate a Hilbert space $H(E)$, which consists of all entire functions f such that both $f(z)/E(z)$ and $f^*(z)/E(z)$ belong to H^2 of the upper half-plane; the $H(E)$-norm of f is given by

$$\|f\|_E^2 = \int_{-\infty}^{\infty} \frac{|f(t)|^2}{|E(t)|^2}\, dx.$$

It is clear that $H(E)$ is a de Branges space. The following fundamental theorem of de Branges says that all de Branges spaces arise in this way [dB68, p. 57].

Theorem C. *A Hilbert space H whose elements are entire functions, which satisfies* (H1), (H2), *and* (H3), *and which contains a nonzero element, is equal isometrically to some space $H(E)$.*

It follows from condition (H2) that for each nonreal ζ there exists a reproducing kernel $K(\zeta, z)$ for the space $H(E)$. The kernel has the following representation, which also defines it for real ζ [dB68, p. 50].

Theorem D. *For each $\zeta \in \mathbb{C}$ the function*

$$K_E(\zeta, z) = \frac{i}{2}\,\frac{E(z)\overline{E(\zeta)} - E^*(z)\overline{E^*(\zeta)}}{\pi(z-\bar{\zeta})}, \tag{1}$$

considered as a function of z, belongs to $H(E)$ and it is the reproducing kernel of $H(E)$, i.e.

$$\langle f, K_E(\zeta,\cdot)\rangle_E = \int_{-\infty}^{\infty} \frac{f(t)\overline{K_E(\zeta,t)}}{|E(t)|^2}\, dt = f(\zeta),$$

for each $f \in H(E)$.

For $x \in \mathbb{R}$, write $E(x)$ as

$$E(x) = |E(x)|e^{-i\varphi(x)},$$

where $\varphi(x)$ is some continuous function in $\mathbb{R}$ such that $E(x)e^{i\varphi(x)}$ is real for all $x \in \mathbb{R}$. If $E(x) \neq 0$, then (1) yields

$$\|K_E(x,\cdot)\|_E^2 = K_E(x,x) = \frac{1}{\pi}\varphi'(x)|E(x)|^2. \tag{2}$$

We say that φ is a *phase function* of E. If there is a need to distinguish φ from phase functions of other functions in $\overline{\mathrm{HB}}$, then we set $\varphi = \varphi_E$.

We shall make use of the following remarkable extension of the Plancherel identity [dB68, p. 55].

THEOREM E. *Let $H(E)$ be a de Branges space and φ a phase function associated with E. Suppose α is a real number and let $\Gamma = \{\gamma_k\}$ be the sequence of real numbers such that $\varphi(\gamma_k) = \alpha + k\pi$, $k \in \mathbb{Z}$. Then if $e^{i\alpha}E - e^{-i\alpha}E^* \notin H(E)$, then the normalized reproducing kernels $K_E(\gamma_k, z)/\|K_E(\gamma_k,\cdot)\|_E$ constitute an orthonormal basis for $H(E)$. In particular,*

$$\|f\|_E^2 = \sum_k \frac{\pi|f(\gamma_k)|^2}{\varphi'(\gamma_k)|E(\gamma_k)|^2}$$

for all $f \in H(E)$; $e^{i\alpha}E - e^{-i\alpha}E^* \in H(E)$ holds for at most one α, modulo π.

The following theorem will also be crucial [dB60]:

THEOREM F. *Let μ be a nonnegative measure on $\mathbb{R}$ and E some function belonging to $\overline{\mathrm{HB}}$. Then*

$$\int_{\mathbb{R}} |f(t)/E(t)|^2\, d\mu(t) = \int_{\mathbb{R}} |f(t)/E(t)|^2\, dt$$

for all $f \in H(E)$ if and only if there exists a bounded holomorphic function A in the upper half-plane $\mathbb{C}^+$ such that $\|A\|_\infty = \sup_{z\in\mathbb{C}^+}|A(z)| \leq 1$ and

$$\frac{y}{\pi}\int_{\mathbb{R}} \frac{d\mu(t)}{(t-x)^2+y^2} = \mathrm{Re}\,\frac{E+E^*A}{E-E^*A}.$$

We write $H(E) = H(F)$ if $H(E)$ and $H(F)$ coincide when considered as sets and the norm of $H(F)$ is equivalent to the norm of $H(E)$. The functions $E \in \overline{\mathrm{HB}}$ with the property that $\mathrm{PW} = H(E)$ have been characterized in [LS99].

3. Main theorem

If Λ is a sampling sequence, then there exists a separated subsequence Λ' which is also sampling (cf. Lemma 3.11 of [Se95]). Hence without loss of generality, we may restrict our attention to separated sequences Λ.

THEOREM 1. *A separated sequence Λ of real numbers is sampling for* PW *if and only if there exist two entire functions $E, F \in \overline{\mathrm{HB}}$ such that*

(i) $H(E) = \mathrm{PW}$,

(ii) Λ *constitutes the zero sequence of* $EF + E^*F^*$.

The following notation will be used repeatedly below: We write $f \lesssim g$ if there is a constant K such that $f \le Kg$; we write $f \simeq g$ if both $f \lesssim g$ and $g \lesssim f$.

Proof. Let us assume first that Λ is a sampling sequence. This means that the Paley-Wiener space equipped with the norm $\sqrt{\sum_k |f(\lambda_k)|^2}$ is a de Branges space. Therefore, Theorem C provides us with a function $E \in \overline{\mathrm{HB}}$ such that $H(E) = \mathrm{PW}$ and

$$\sum_k |f(\lambda_k)|^2 = \int_{\mathbb{R}} \frac{|f(t)|^2}{|E(t)|^2}\, dt.$$

Since the measure $\mu = \sum_k |E(\lambda_k)|^2 \delta_{\lambda_k}$ meets the hypothesis of Theorem F, there exists a bounded holomorphic function A in the upper half-plane with norm $\|A\|_\infty \le 1$ and a real number a such that

$$-i \sum_k |E(\lambda_k)|^2 \left(\frac{1}{z - \lambda_k} + \frac{1}{\lambda_k} \right) + ia = \frac{E(z) + E^*(z)A(z)}{E(z) - E^*(z)A(z)}. \tag{3}$$

Note that the right-hand side is a holomorphic function defined in the upper half-plane, but the left-hand side is a meromorphic function defined in the whole plane. We denote this meromorphic function by M. The following relationship holds when $\operatorname{Im} z > 0$:

$$A = \frac{M-1}{M+1} \frac{E}{E^*}.$$

The function $M - 1$ has poles at the points λ_k as we can see from (3), and it vanishes whenever E^* vanishes. Setting

$$G(z) = \prod_k \left(1 - \frac{z}{\lambda_k}\right) e^{z/\lambda_k},$$

we may therefore write

$$M - 1 = -\frac{E^* F^*}{G},$$

with F an entire function. We also have $M^* = -M$ and $G^* = G$, and consequently $M + 1 = EF/G$. It follows that $F^*/F = -A$ in the upper half-plane and F has no zeros in $\operatorname{Im} z > 0$. Since $|A|$ is bounded by 1, it follows that $F \in \overline{\mathrm{HB}}$.

We will now see that $G = (EF + E^*F^*)/2$, which means that Λ is the zero sequence of $EF + E^*F^*$. We know that $G = -MG + EF$. If $x \in \mathbb{R}$, then $G(x)$ is real and $M(x)G(x)$ is an imaginary number. Thus G is the real part of EF for real z, whence $G(z) = (EF + E^*F^*)/2$ for all $z \in \mathbb{C}$.

We next prove the converse implication. The function E has no zeros on the real axis since $\mathrm{PW} = H(E)$. We may assume that F has no real zeros. If this were not the case, then $F = S\widetilde{F}$, where $\widetilde{F} \in \overline{\mathrm{HB}}$ without real zeros and S is an entire function, with real zero set $Z(S)$, and moreover $S^* = S$. Therefore $\Lambda = \Lambda_1 \cup Z(S)$ and Λ_1 will be the zero set of $E\widetilde{F} + E^*\widetilde{F}^*$. If we prove that Λ_1 is a sampling sequence, then Λ is a also a sampling sequence.

Given $\alpha \in (0, \pi]$ we let Λ_α be the sequence of points $\lambda_{\alpha,k}$ such that $\varphi_{EF}(\lambda_{\alpha,k}) = \alpha + k\pi$, $k \in \mathbb{Z}$. Observe that since Λ is the zero sequence of $EF + E^*F^*$, and E and F have no real zeros, we have $\Lambda = \Lambda_{\pi/2}$. For $\alpha \neq \pi/2$ the sequence Λ_α is interlaced with the sequence Λ. Hence since Λ is a separated sequence, Λ_α can be expressed as the union of two separated sequences, each with separation constant not smaller than that of Λ. Thus the Plancherel-Pólya inequality [Yo80, pp. 96–98] implies that

$$\sum_k |f(\lambda_{\alpha,k})|^2 \le C\|f\|_{\mathrm{PW}}^2, \tag{4}$$

with C independent of α. Moreover, for all values of α except possibly one, Theorem E implies that for every $g \in H(EF)$ the following relation holds:

$$\int_{\mathbb{R}} \frac{|g(t)|^2}{|E(t)F(t)|^2}\,dt = \sum_k \frac{|g(\lambda_{\alpha,k})|^2}{|E(\lambda_{\alpha,k})|^2|F(\lambda_{\alpha,k})|^2\varphi'_{EF}(\lambda_{\alpha,k})}.$$

For every $f \in H(E)$ the function $g = Ff$ belongs to $H(EF)$ with $\|f\|_{H(E)} = \|Ff\|_{H(EF)}$. Therefore, for every $f \in \mathrm{PW}$ we have

$$\|f\|_{\mathrm{PW}}^2 \simeq \int_{\mathbb{R}} \left|\frac{f(t)}{E(t)}\right|^2 dt = \int_{\mathbb{R}} \left|\frac{g(t)}{E(t)F(t)}\right|^2 dt = \sum_k \frac{|f(\lambda_{\alpha,k})|^2}{|E(\lambda_{\alpha,k})|^2\varphi'_{EF}(\lambda_{\alpha,k})}. \tag{5}$$

Since $H(E) = \mathrm{PW}$, E does not vanish for $x \in \mathbb{R}$ and therefore by (2)

$$1 = \sup_{\substack{f\in \mathrm{PW}\\ \|f\|_{\mathrm{PW}}^2 \le 1}} |f(x)|^2 \simeq \sup_{\substack{f\in H(E)\\ \|f\|_E^2 \le 1}} |f(x)| = K_E(x,x) = \frac{1}{\pi}\varphi'_E(x)|E(x)|^2$$

for $x \in \mathbb{R}$. Since $\varphi'_{EF} = \varphi'_E + \varphi'_F \ge \varphi'_E$, we obtain from (5) the inequality

$$\|f\|_{\mathrm{PW}}^2 \le C \sum_k |f(\lambda_{\alpha,k})|^2,$$

where the constant C does not depend on α. This inequality may fail for one $\alpha \in [0, \pi)$. We may assume it fails for $\alpha = \pi/2$, because otherwise we have proved that Λ is sampling. Take a sequence $\alpha_n \to \pi/2$. Then $\lambda_{\alpha_n,k} \to \lambda_k$ for all k, and by (4) we may apply Lebesgue's dominated convergence theorem to conclude that the inequality holds also for $\alpha = \pi/2$. □

If Λ is a complete interpolating sequence, then there exists an $E \in \overline{\mathrm{HB}}$ such that $H(E) = \mathrm{PW}$ and Λ constitutes the zero sequence of $E + E^*$. This follows from Theorems C and E. Therefore, we may view the function F of Theorem 1 as accounting for the "redundancy" in Λ. There is a trivial case in which this is particularly transparent: Suppose $\Lambda = \Lambda' \cup (\Lambda \setminus \Lambda')$, where Λ' is a complete interpolating sequence. (Such a decomposition holds whenever $D^-(\Lambda) > 1$; cf. Theorem 2.3 of [Se95].) Then the condition of Theorem 1 is met if we choose F to be the generating function of the "redundant" sequence $\Lambda \setminus \Lambda'$, i.e., if we set

$$F(z) = \prod_{\lambda_k \in \Lambda \setminus \Lambda'} \left(1 - \frac{z}{\lambda_k}\right) e^{z/\lambda_k}.$$

A somewhat different way of seeing F as measuring the "redundancy" in Λ is as follows. The notion of a complete interpolating sequence may be extended in a natural way: We say that Λ is a complete interpolating sequence for a de Branges space H if the interpolation problem $f(\lambda_k) = a_k$ has a unique solution $f \in H$ whenever

$$\sum_k \frac{|a_k|^2}{K(\lambda_k, \lambda_k)} < \infty.$$

Then Theorem 1 says that if Λ is sampling for PW, then Λ is a complete interpolating sequence for $H(EF)$, and $H(E)$ (with $H(E) = \mathrm{PW}$) is isometrically embedded into $H(EF)$ by the map $f \to Ff$. This interpretation of Theorem 1 has an interesting relation to the result of [Se95] which says that we cannot in general obtain from Λ a complete interpolating sequence for PW by making *the sequence thinner*: Instead, we can make *the space bigger* so that Λ becomes a complete interpolating sequence for the bigger space.

We note in passing that Theorem 1 solves the following problem of norm equivalence for PW: Which nonnegative measures μ on $\mathbb{R}$ have the property that the $L^2(\mathbb{R}, d\mu)$-norm yields a norm for PW equivalent to the standard $L^2(\mathbb{R})$-norm? To apply Theorem 1 to this problem, we need a way of associating sampling sequences with such measures. Given a nonnegative measure μ and two positive numbers r, δ, define

$$\Lambda_\mu(\delta, r) = \{kr : \ k \in \mathbb{Z} \text{ and } \mu([kr, (k+1)r)) \geq \delta\}.$$

Then the Bernstein and the Plancherel-Pólya inequalities imply (see [Or98]):

Proposition. *The $L^2(\mathbb{R},\mu)$-norm yields an equivalent norm for* PW *if and only if the following holds*:

(i) *There exists a positive constant C such that $\mu([x,x+1)) \le C$ for all $x \in \mathbb{R}$.*

(ii) *For all sufficiently small $r > 0$ there exists a $\delta = \delta(r) > 0$ such that $\Lambda_\mu(r,\delta)$ is sampling for* PW.

Problems about norm equivalence for spaces of entire functions of exponential type in one and several variables have been studied by many authors. See, e.g., [Li65] and [LLS92] for an extensive historical account.

4. Landau's inequality and the John-Nirenberg theorem

The following corollary says that a sampling sequence is "everywhere denser" than some complete interpolating sequence.

Corollary 1. *If Λ is a separated sampling sequence for* PW, *then there exists a complete interpolating sequence $\Gamma = \{\gamma_k\}_{k=-\infty}^{\infty}$ such that for every $k \in \mathbb{Z}$ there is at least one point $\lambda \in \Lambda$ such that $\gamma_k \le \lambda < \gamma_{k+1}$.*

Proof. If Λ is a sampling sequence, then Λ consists of those points such that $\varphi_{EF}(\lambda) = \pi/2 + k\pi$ for some $k \in \mathbb{Z}$. On the other hand, φ_E is an increasing function which grows more slowly than φ_{EF}. Thus the sequence Γ_α which consists of those points such that $\varphi_E(\gamma) = \alpha + k\pi$ for some $k \in \mathbb{Z}$, has the claimed property. Since $H(E) = \mathrm{PW}$, Theorem E implies that Γ_α is a complete interpolating sequence for all values of α except possibly one, mod π. Since the choice of α is at our disposal, the proof is complete. □

Let us return to Landau's inequality, which is stated as Theorem B above. To begin with, let us show by an example that the inequality is best possible. Set $\Lambda = \{k + \log^+|k|\}_{k\in\mathbb{Z}}$. This sequence is sampling, because the function

$$F(z) = \lim_{R\to\infty} \prod_{|\lambda_k|<R} (1 - z/\lambda_k)$$

is a sine-type function; i.e., $|F(z)|e^{-\pi|\operatorname{Im} z|} \simeq 1$ for $|\operatorname{Im} z| > 1$. Thus a classical theorem of Levin [HNP81, p. 250] implies that Λ is a complete interpolating sequence. (Alternatively, note that this means $v = 0$ in Theorem A.) On the other hand, it is clear that $n_\Lambda(R) = R - \log R + O(1)$ and $-n_\Lambda(-R) = R + \log R + O(1)$ when $R \to \infty$.

Claim. *Landau's inequality is a consequence of the John-Nirenberg theorem for* BMO *functions.*

Proof. Suppose Λ is a sampling sequence, and let Γ be an associated complete interpolating sequence as described in Corollary 1. By Corollary 1 and Theorem A,

$$n_\Lambda(b) - n_\Lambda(a) \geq n_\Gamma(b) - n_\Gamma(a) - 1 \geq b - a + h(b) - h(a) - 1,$$

where $h \in \mathrm{BMO}$ of a special form. (For this proof we will only need the fact that $h \in \mathrm{BMO}$.) Set $I = [a, b]$. Then the triangle inequality in the form

$$|h(b) - h(a)| \leq |h(b) - h_I| + |h(a) - h_I|,$$

gives

$$n_\Lambda(b) - n_\Lambda(a) \geq b - a - 2 \max_{t \in I} |h(t) - h_I| - 1. \tag{6}$$

Define

$$J = \{t \in I; |h(t) - h_I| \geq A \log |I|\}.$$

Then the John-Nirenberg theorem (cf. [JN61], [Ga81, p. 230]) implies

$$|J| \leq C e^{-\frac{cA \log |I|}{\|h\|_*}} |I| \leq C/|I|$$

for some sufficiently big A, with C, c absolute positive constants. But $h'(t)$ is bounded, in fact equal to -1 for $t \notin \Lambda$, and so if the right-hand side is smaller than the separation distance of Λ, then $|h(t) - h_I| \leq A \log |I| + B$ for all $t \in I$. Plugging this into (6), we obtain Landau's inequality. $\square$

It should be stressed that we do not pretend to have given an easier proof of Landaus's inequality, as the John-Nirenberg theorem is admittedly a more sophisticated result. However, we find the link between the two results quite intriguing.

5. Sampling sequences and approximation of subharmonic functions

In Section 3, we explained how Theorem 1 could be interpreted in the trivial case in which there exists a subsequence $\Lambda' \subset \Lambda$ which is a complete interpolating sequence. The purpose of this section is to demonstrate the applicability of our condition when $D^-(\Lambda) = 1$ and no such subsequence Λ' exists.

To obtain positive results from Theorem 1, one needs information about those $E \in \overline{\mathrm{HB}}$ for which $H(E) = \mathrm{PW}$. As mentioned at the end of Section 2, such functions are described in [LS99]. However, we will not use this description, which involves a rather delicate statement about the location of the zeros of E. In fact, we will obtain precise results by using only the evident fact that $H(E) = \mathrm{PW}$ if $|E(z)| \simeq e^{\pi \operatorname{Im} z}$ for $\operatorname{Im} z \geq 0$.

We will assume $\psi \in \mathcal{C}^1(\mathbb{R})$ is a nondecreasing function satisfying $\psi(\infty) - \psi(-\infty) = \infty$ and $\psi'(x) = o(1)$ as $|x| \to \infty$. We associate with ψ a sequence $\Lambda(\psi) = \{\lambda_k\}_{k\in\mathbb{Z}}$ defined by $\lambda_k + \psi(\lambda_k) = k$. In other words, with $[x]$ denoting the integer part of the real number x and $\psi(0) = 0$, this means that $n_{\Lambda(\psi)}(t) = [t + \psi(t)]$. All sequences $\Lambda(\psi)$ are sets of uniqueness of infinite excess, but none of them contain subsequences which are complete interpolating sequences; cf. Theorem 2.7 in [Se95]. We introduce the potential

$$U_\psi(z) = \int_{-\infty}^{\infty} [\log|1 - z/t| + \operatorname{Re} z/t]\, d\psi(t),$$

with the integral taken in the principal value sense. This function is subharmonic because $\psi'(t) \geq 0$.

We will now demonstrate how our sampling condition is related to the problem of approximating U_ψ by $\log|f|$, with f an entire function.

Corollary 2. *A sequence $\Lambda(\psi)$ is sampling for* PW *if there exists an $f \in \overline{\mathrm{HB}}$ such that $\varphi_f'(x) = o(1)$ when $|x| \to \infty$ and*

$$|U_\psi(z) - \log|f(z)|| \lesssim 1 \quad \textit{for} \quad \operatorname{Im} z \geq 0. \tag{7}$$

Proof. If we could find $e \in \overline{\mathrm{HB}}$ such that

$$\varphi_e(x) = \pi x + \pi\psi(x) - \varphi_f(x),$$

we would be done, because then Λ would constitute the zero sequence of $ef + e^* f^*$, and $|e(z)| \simeq e^{\pi \operatorname{Im} z}$ for $\operatorname{Im} z \geq 0$. In general, however, it is impossible to find such an e. Instead, we will appeal to the following perturbation argument: If a separated sequence $\Gamma = \{\gamma_k\}$ is sampling, then $\Gamma' = \{\gamma_k + \delta_k\}$ is sampling whenever all $\gamma_k + \delta_k$ are distinct and $\delta_k \to 0$ as $|k| \to \infty$. (This follows from Lemma III in [DS52] and the elementary fact that Γ is sampling if $\delta_k = 0$ except for finitely many integers k.) Thus it is enough to construct an $E \in \overline{\mathrm{HB}}$ such that

$$\varphi_E(x) - \pi x - \pi\psi(x) + \varphi_f(x) = o(1) \tag{8}$$

when $|x| \to \infty$ and $|E(z)| \simeq e^{\pi \operatorname{Im} z}$ for $\operatorname{Im} z \geq 0$, because then the perturbation argument applies with the zero sequence of $Ef + E^* f^*$ playing the role of Γ and $\Gamma' = \Lambda$.

We may assume the function $\omega(x) = x + \psi(x) - \varphi_f(x)/\pi$ satisfies $\omega'(x) \simeq 1$. Partition the real line into a sequence of disjoint intervals $I_k = [x_k, x_{k+1}]$, $k \in \mathbb{Z}$, with $x_0 = 0$, such that

$$\int_{I_k} \omega'(t)dt = 1$$

for all k, and choose $\gamma_k \in I_k$ so that

$$\gamma_k = \int_{I_k} t\omega'(t)dt.$$

We set

$$A(z) = \lim_{R\to\infty} \prod_{|\gamma_k|<R} (1 - z/\gamma_k)$$

and $\Gamma = \{\gamma_k\}$. Then by Lemma 3 in [OS99],

$$|A(z)|e^{-U_\omega(z)} \simeq \min(1, \mathrm{dist}(z, \Gamma)).$$

Now choose two monomials P and Q of the same degree and with only real zeros such that $B(z) = A(z - 1/2)P(z)/Q(z)$ is an entire function and the zeros of A and B are interlaced. Then according to a theorem of Meĭman (see [Le80, p. 314]), we have either $A - iB \in \overline{\mathrm{HB}}$ or $A + iB \in \overline{\mathrm{HB}}$; we may set $E = A - iB$ and assume $E \in \overline{\mathrm{HB}}$ because P may be replaced by $-P$.

It remains to show that E satisfies (8). Consider the sequence of functions $A_k(z) = A(z - x_{2k})$. A normal family argument shows that there exists a sequence $c_k \simeq 1$ such that $A_k(x) - c_k \cos \pi x \to 0$ uniformly on compact subsets of the real line. Similarly, if we set $B_k(z) = B(z - x_{2k})$, we obtain that $B_k(x) - c_k \sin \pi x \to 0$ uniformly on compact subsets of the real line. Then it follows from the construction of E that

$$E(x) = |E(x)|e^{i(\pi(x+\psi(x)) - \varphi_f(x) + o(1))}$$

when $x \to \infty$, with $|E(x)| \simeq 1$. □

To solve the approximation problem for U_ψ, we shall adapt an approach from a recent paper by Lyubarskii and Malinnikova [LM01], which contains rather conclusive results about the possibility of approximating an arbitrary subharmonic function by $\log |f|$ with f an entire function.

To ease the exposition, we set $\psi(x) = 0$ for $x \le 0$. (It is not difficult to modify the construction below to get a corresponding result when both $\psi(\infty) = \infty$ and $\psi(-\infty) = -\infty$.) Let $\{t_n\}_{n=0}^{\infty}$ be the sequence such that $t_0 = 0$ and $\psi(t_n) = n$, $n = 1, 2, 3, ...$; set $d_n = t_n - t_{n-1}$. We will say that ψ induces a *logarithmically regular partition* if $d_n \simeq d_{n+1}$ and

$$\sup_{x>0} \sum_{x/2<t_n<2x} \frac{d_n^2}{(x - t_n)^2 + d_n^2} < \infty.$$

Theorem 2. *Suppose $\psi(x) = 0$ for $x \le 0$. Then*

(i) *If $\psi'(x) = 1/O(x)$ when $x \to \infty$, and ψ induces a logarithmically regular partition, then $\Lambda(\psi)$ is sampling for* PW.

(ii) *If $\psi'(x) = o(1/x)$ when $x \to \infty$, then $\Lambda(\psi)$ is not sampling for* PW.

We note that $\psi(x) = \sqrt{x}$ for $x \geq 0$ corresponds to the example of [Se95] showing the existence of a sampling sequence containing no subsequence which is a complete interpolating sequence.

Case (i) is proved by means of Corollary 2, while case (ii) is proved by a direct argument. The point of case (ii) is to show that the growth condition $\psi'(x) = 1/O(x)$ is critical. Thus Theorem 2 indicates that the condition of Corollary 2 is in a sense close to being necessary.

Proof. We consider first statement (i). According to Corollary 2, we need to show that the assumptions on ψ ensure the existence of a function $f \in \overline{\mathrm{HB}}$ such that (7) holds when $\operatorname{Im} z \geq 0$.

The zeros of f are determined as follows. Define $r_n \in (t_{n-1}, t_n)$ by

$$\log r_n = \int_{t_{n-1}}^{t_n} \log t \; d\psi(t), \tag{9}$$

and then

$$z_n = r_n e^{-icd_n/r_n}$$

with $c > 0$ so small that $cd_n/r_n \leq \pi/4$ for all n. We choose f in such a way that it satisfies

$$\log|f(z)| = \sum_{n=1}^{\infty} \int_{r_{n-1}}^{r_n} \left(\log\left|1 - \frac{z}{z_n}\right| + \operatorname{Re}\frac{z}{t} \right) d\psi(t).$$

Clearly f belongs to $\overline{\mathrm{HB}}$. We set $V = U_\psi - \log|f|$; it satisfies the following relation:

$$V(z) = \sum_{n=1}^{\infty} \int_{r_{n-1}}^{r_n} \left(\log\left|1 - \frac{z}{t}\right| - \log\left|1 - \frac{z}{z_n}\right| \right) d\psi(t) = \sum_{n=1}^{\infty} j_n(z). \tag{10}$$

We see that it is enough to prove that the series converges uniformly on compact sets in $\mathbb{C}$, and that $V(z) = O(1)$ for $\operatorname{Im} z \geq 0$.

The proof is a simplified version of an argument from [LM01]. We shall therefore be brief and refer to [LM01] for further details.

Given $z \in \mathbb{C}$, we let $n(z)$ be a positive integer n such that $r_{n-1} < |z| \leq r_n$. Then if $\operatorname{Im} z \geq 0$, it is plain that the smoothness of ψ ensures that

$$\sum_{n=n(z)-1}^{n(z)+1} j_n(z) \simeq 1.$$

Next let $n^-(z)$ be the positive integer n such that $r_{n-1} < |z|/2 \leq r_n$, and $n^+(z)$ be the positive integer n such that $r_{n-1} < 2|z| \leq r_n$. Then for $z \in \mathbb{C}$ we see that

$$\sum_{n=n^+(z)+1}^{\infty} j_n(z) \simeq 1.$$

After observing that by (9) we may write

$$j_n(z) = \int_{r_{n-1}}^{r_n} \left(\log \left| 1 - \frac{t}{z} \right| - \log \left| 1 - \frac{z_n}{z} \right| \right) d\psi(t),$$

we obtain

$$\sum_{n=1}^{n^-(z)-1} j_n(z) \simeq 1$$

in a similar way. Thus we have in particular established the uniform convergence on compact sets.

We set

$$N(z) = \{n^-(z), n^-(z)+1, ..., n^+(z)-1, n^+(z)\} \setminus \{n(z)-1, n(z), n(z)+1\}$$

(the set of "essential indices") and split the corresponding sum into two parts,

$$\begin{aligned} V_1(z) + V_2(z) \quad &= \quad \sum_{n \in N(z)} \int_{r_{n-1}}^{r_n} \left(\log \left| 1 - \frac{z}{t} \right| - \log \left| 1 - \frac{z}{r_n} \right| \right) d\psi(t) \\ &\quad + \sum_{n \in N(z)} (\log |z - r_n| - \log |z - z_n|) \end{aligned}$$

now assuming that $\operatorname{Im} z \geq 0$.

To estimate V_1, introduce the function

$$L(\omega) = \log(1 - ze^{-\omega}),$$

which for each $n \in N(z)$ is analytic in a domain containing those ω such that $e^\omega \in [t_{n-1}, t_n]$. Therefore,

$$L(\omega) - L(\omega_n) = (\omega - \omega_n) L'(\omega_n) + \int_{\omega_n}^{\omega} L''(\sigma)(\omega - \sigma) d\sigma = (\omega - \omega_n) L'(\omega_n) + Q_n(z, t),$$

where $t = e^\omega \in [t_{n-1}, t_n]$ and $e^{\omega_n} = r_n$. By (9), we obtain

$$V_1(z) = \operatorname{Re} \sum_{n \in N(z)} \int_{t_{n-1}}^{t_n} Q_n(z, t) d\psi(t).$$

A direct estimate gives

$$\sup_{t \in [t_{n-1}, t_n]} |Q_n(z, t)| \lesssim \frac{d_n^2}{|z - r_n|^2}.$$

Using the assumption that ψ induces a logarithmically regular partition, we obtain that $V_1(z) \simeq 1$.

To deal with V_2, we write

$$\log |z - r_n| - \log |z - z_n| = \operatorname{Re} \int_{z_n}^{r_n} \frac{d\zeta}{\zeta - z}.$$

Integrating along the arc $\zeta = r_n e^{-i\theta}$, $0 \le \theta \le c d_n / r_n$, we obtain from this

$$\left| \log |z - r_n| - \log |z - z_n| \right| \lesssim \frac{d_n^2}{|z - r_n|^2}.$$

Using again the assumption that ψ induces a logarithmically regular partition, we have also proved that $V_2(z) \simeq 1$, and are done with the proof of (10).

We consider next statement (ii). Our plan is to construct a sequence of functions $f_n \in \mathrm{PW}$ for which $\sum |f_n(\lambda_k)|^2 / \|f_n\|^2_{\mathrm{PW}} \to 0$.

Let t_n be the sequence such that $\psi(t_n) = n$, $n = 1, 2, 3, ...$, and suppose n is sufficiently big for the following construction to be feasible. We require $\xi_n \in (t_n, t_{n+1}/2)$ to be such that $\psi(\xi_n) = n + 1/2 + \epsilon_n$, where ϵ_n will be chosen below. Define a bounded, continuous function ϕ_n such that $\phi_n(t) = -t$ when $|t| < 1/2$, $\phi_n(t) = \psi_n(t) - n - 1/2$ for $t_n < t < \xi_n$ and $\phi_n(t) = \psi(t) - n - 3/2$ for $2\xi_n < t < t_{n+1}$, and otherwise ϕ_n is linear. We choose ϵ_n such that

$$\int_{t_n}^{\xi_n} \frac{\phi_n(x)}{x} dx = 0.$$

It is clear that $\epsilon_n \to 0$ when $n \to \infty$.

Define a subharmonic function U_n by

$$U_n(z) = \lim_{R \to \infty} \int_{-R}^{R} [\log |1 - z/t|](1 + \phi_n'(t)) dt.$$

A direct computation gives us the estimate

$$U_n(x) = \int_0^{|x|} \frac{\phi_n(t) - 1/2}{t} dt + O(1),$$

when $|x| \to \infty$. From this and the assumption $\phi_n'(t) = o(1/t)$, we see that there exists an interval $[(1 - o(1))\xi_n, 2\xi_n]$ such that $U_n(x) + (1/2)\log x \simeq 1$ when $|x| \in [(1 - o(1))\xi_n, 2\xi_n]$. On the other hand, for $|x| \notin [t_n, t_{n+1}]$ we have $U_n(x) = -\log |x| + O(1)$. Setting $\Omega_n = [t_n, \xi_n] \cup [2\xi_n, t_{n+1}]$, we observe that

$$\int_{\Omega_n} e^{2U_n(x)} dx \to \infty, \tag{11}$$

but

$$\int_{\mathbb{R} \setminus \Omega_n} e^{2U_n(x)} dx \lesssim 1. \tag{12}$$

Thus e^{U_n} belongs to $L^2(\mathbb{R})$, but its L^2-norm tends to ∞. Likewise, if a sequence of real numbers $\Gamma = \{\gamma_k\}_{k=-\infty}^{\infty}$ satisfies $\gamma_{k+1} - \gamma_k \simeq 1$, we have

$$\sum_{\gamma \in \mathbb{R} \setminus \Omega_n} e^{2U_n(\gamma_k)} \lesssim 1. \tag{13}$$

The construction of f_n is now identical to the construction of the function A in the proof of Corollary 2, with ω' replaced by $\phi_n' + 1$. Thus we obtain

$$f_n(z) = \lim_{R\to\infty} \prod_{|\gamma_k|<R} (1 - z/\gamma_k)$$

with $\Gamma = \{\gamma_k\}$ a separated real sequence and such that

$$|f_n(z)|e^{-U_n(z)} \simeq \min(1, \operatorname{dist}(z, \Gamma)).$$

This f_n is of exponential type π. By (12), $f_n \in \mathrm{PW}$, but also $\|f_n\|_{\mathrm{PW}} \to \infty$, in view of (11).

Observe that $\operatorname{dist}(\lambda_k, \Gamma) \to 0$ uniformly for $\lambda_k \in \Omega_n$, when $n \to \infty$, and that $\gamma_k = k$ for all $k < 0$. Along with (13) and our estimates for f_n, this implies that $\sum_k |f(\lambda_k)|^2/\|f\|^2_{\mathrm{PW}} \to 0$, so that Λ is not sampling. $\square$

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References

[dB60] L. de Branges, Some Hilbert spaces of entire functions, *Trans. A. M. S.* **96** (1960), 259–295.

[dB68] ———, *Hilbert Spaces of Entire Functions*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1968.

[Da92] I. Daubechies, *Ten Lectures on Wavelets*, *CBMS-NSF Regional Conference Series in Applied Mathematics*, **61**, SIAM, Philadelphia, PA, 1992.

[DS52] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, *Trans. A. M. S.* **72** (1952), 341–366.

[Ga81] J. B. Garnett, *Bounded Analytic Functions*, *Pure and Applied Math.* **96**, Academic Press, New York, 1981.

[HNP81] S. V. Hruščev, N. K. Nikol'skiĭ, and B. S. Pavlov, Unconditional bases of exponentials and reproducing kernels, in *Complex Analysis and Spectral Theory* (V. P. Havin and N. K. Nikol'skiĭ, eds.), *Lecture Notes in Mathematics* **864**, 214–335, Springer-Verlag, New York, 1981.

[Ja91] S. Jaffard, A density criterion for frames of complex exponentials, *Michigan Math. J.* **38** (1991), 339–348.

[JN61] F. John and L. Nirenberg, On functions of bounded mean oscillation, *Comm. Pure Appl. Math.* **14** (1961), 415–426.

[La67] H. J. Landau, Necessary density conditions for sampling and interpolation of certain entire functions, *Acta Math.* **117** (1967), 37–52.

[Le80] B. Ya. Levin, *Distribution of Zeros of Entire Functions*, *Transl. Math. Monographs* **5**, A. M. S., Providence R.I., 1980.

[LLS92] B. Ya. Levin, V. N. Logvinenko, and M. L. Sodin, Subharmonic functions of finite degree bounded on subsets of the "real hyperplane", in *Entire and Subharmonic Functions*, *Adv. Soviet Math.* **11**, A. M. S., Providence, RI, 1992.

[Li65] V. Ja. Lin, On equivalent norms in the space of square integrable entire functions of exponential type, *Mat. Sb. (N.S.)* **67** (1965), 586–608 (Russian).

[LM01] Yu. Lyubarskii and E. Malinnikova, On approximation of subharmonic functions, *J. Analyse Math.* **83** (2001), 121–149.

[LS97] Yu. Lyubarskii and K. Seip, Complete interpolating sequences for Paley-Wiener spaces and Muckenhoupt's (A_p) condition, *Rev. Mat. Iberoamericana* **13** (1997), 361–376.

[LS99] ———, Weighted Paley-Wiener spaces, preprint, 1999.

[Or98] J. Ortega-Cerdà, Sampling measures, *Publ. Mat.* **42** (1998), 559–566.

[OS99] J. Ortega-Cerdà and K. Seip, Multipliers for entire functions and an interpolation problem of Beurling, *J. Funct. Anal.* **162** (1999), 400–415.

[Pa79] B. S. Pavlov, The basis property of a system of exponentials and the condition of Muckenhoupt, *Dokl. Akad. Nauk SSSR* **247** (1979), 37–40.

[Se95] K. Seip, On the connection between exponential bases and certain related sequences in $L^2(-\pi, \pi)$, *J. Funct. Anal.* **130** (1995), 131–160.

[Yo80] R. M. Young, *An Introduction to Nonharmonic Fourier Series*, *Pure and Applied Math.* **93**, Academic Press, New York, 1980.

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